Referring to Fig. 2, the unit tangent to the vortex line is

$$\underline{\tau} = (\underline{n}_0 \times \underline{n}_I)/|\underline{n}_0 \times \underline{n}_I| \tag{6}$$

where $\underline{n}_0$ and $\underline{n}_I$ are known, respectively, from the geometry of the parabolic cylinder and vortex sheet at P′, the point in question. The unit downstream normal to the vortex line, which is also tangential to the sheet, is

$$\underline{n}_2 = \underline{\tau} \times \underline{n}_I \tag{7}$$

and the sheet vorticity vector at P′ is given by $\omega\underline{\tau}$, where

$$\omega = \frac{\mathrm{d}\psi}{\mathrm{d}\xi}\,\frac{\mathrm{d}\zeta}{\mathrm{d}n_2} \tag{8}$$

and $\psi(\xi)$ is a scalar vorticity-flux function representing the vorticity flowing onto the sheet across the separation line, up to $P(\xi,\eta,\zeta)$, i.e.,

$$\psi(\xi) = \int_{\xi_0}^{\xi} [\omega(\underline{n}_2 \cdot \underline{a}_I)/\ell_I]\,\mathrm{d}x \tag{9}$$

A table of values for ψ may be constructed from the J filament strengths calculated by the VFIT, so that $\mathrm{d}\psi/\mathrm{d}\xi$ can be found numerically. The ξ corresponding to the vortex line through P′ is found by identifying the particular parabolic cylinder on which it lies.

Corrected Velocity Field on the Body near the Separation Line

By replacing the field of the two rows of filament segments nearest to the body by the field of the reconstructed "near" sheet, found using the Biot-Savart law,[13] the required corrected field is obtained, since "distant" parts of the sheet are still adequately represented by the filaments.

Conclusion

A method of calculating laminar separation lines on general smooth aerodynamic bodies is proposed, based on adaptation of an existing slender-body method.[4] The fundamental assumption of the method remains to be checked through suitable test cases, but certain aspects of feasibility and accuracy have been checked and found to be satisfactory.

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Transonic Small-Disturbance Theory for Dusty Gases

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I. Introduction

THE flow of gases containing small particles (dust) is interesting to study and is important in many practical situations. A method for analyzing the effect of the dust on flows in the transonic regime is presented in this Note.

Marble[1] derived a model in which the gas and dust exchange heat and momentum. The equations of conservation of mass, momentum, and energy for each material are simplified by assuming low volumetric concentations of a relatively heavy dust. This set of dusty gas equations is then further approximated by assuming strong coupling between the materials. The resulting system of equations is analogous to the equations for the adiabatic motion of an ideal gas, except that the ratio of specific heats γ and density ρ are modified to reflect the heat capacity and density of the dust. Marble also showed that this generalized gas supports discontinuities in properties (generalized shocks) which consist of a shock in the gas, followed by a relaxation back to equilibrium (both thermal and mechanical) of the dust particles.

Marble's model is applied to transonic small-disturbance theory. It is shown that the effect of the dust is to modify the flow in such a way that it is equivalent to the flow of a clear gas at a different freestream speed around an airfoil of a different thickness.

This work is complementary to that of Barron and Wiley,[2] who derived a version of small-disturbance theory for hypersonic dusty gases. Applying the Newtonian approximation $\gamma \to 1$ and $M \to \infty$, they are able to work out an interesting solution for wedge-shaped bodies. The present work does not use this approximation; instead, a similarity transformation is employed which relates the equations of dusty gases corresponding to those for dust-free flow.

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II. The Equivalent Gas Model

Marble[1] gives the steady equations of motion for a dusty gas in the small-slip approximation as

$$\nabla \cdot \rho_m q = 0 \tag{1}$$

$$\rho_m q \cdot \nabla q + \nabla p = 0 \tag{2}$$

$$\rho_m c_m q \cdot \nabla T - q \cdot \nabla p = 0 \tag{3}$$

where q is the velocity, p the pressure, T the temperature, and

$$\rho_m = \rho_g (1+\kappa) \tag{4}$$

$$c_m = (c_p + \kappa c)/(1+\kappa) \tag{5}$$

are the mixture density and effective specific heat. Here ρ_g is the gas density and $\kappa = \alpha\rho_p/\rho_g$ is the loading. The particle density ρ_p is assumed constant. The volumetric concentration of the dust, α, is shown by Marble to such that κ is constant. Also, c_p is the specific heat at constant pressure for the gas, and c the specific heat for the particles. The equation of state for the gas is taken to be

$$p = R\rho_g T \tag{6}$$

where R is the gas constant.

Combining Eqs. (3) and (6) leads to

$$p = p_0 (\rho_m/\rho_0)^{\hat{\gamma}} \tag{7}$$

where p_0 and ρ_0 are reference constants, and $\hat{\gamma}$ the effective ratio of specific heats for the mixture given by

$$\hat{\gamma} = (c_p + \kappa c)/(c_v + \kappa c) \tag{8}$$

where c_v is the specific heat at constant volume for the gas.

One effect of the dust can be seen immediately. If the local speed of sound in a clear gas is

$$a = \left(\frac{\partial p}{\partial \rho}\right)^{1/2} = \left(\frac{\gamma p}{\rho}\right)^{1/2}$$

the speed of sound in the same gas (at the same thermodynamic state) with dust added is

$$\hat{a} = \left(\frac{\partial p}{\partial \rho_m}\right)^{1/2} = \left(\frac{\hat{\gamma} p}{\rho_m}\right)^{1/2}$$

III. Small-Disturbance Theory

Assuming potential flow, it is possible to formulate a small-disturbance theory for transonic thin airfoils in dusty gases, analogous to that given in Ref. 3. The transonic similarity parameter for this case is a function of the new value of γ. It will be shown how this modified similarity parameter is related to the one for the case of no dust, so that a method of estimating the effect of the dust on the usual small-disturbance theory may be achieved.

The small-disturbance boundary-value problem is based on a limit process expansion for the full potential Φ, so that $\delta \to 0$ with x, $\tilde{y} = \delta^{1/3} y$, and $\hat{K} = (1-\hat{M}_\infty^2)/\delta^{2/3}$ are fixed as $\hat{M}_\infty \to 1$. Here δ is the thickness ratio of the wing, and $\hat{M}_\infty = U/\hat{a}_\infty$ is the freestream Mach number of the dusty gas for freestream speed U. See Fig. 1. The expansion is

$$\Phi(x,y;\delta) = U_\infty \{x + \delta^{2/3}\phi(x,\tilde{y}) + \ldots\} \tag{9}$$

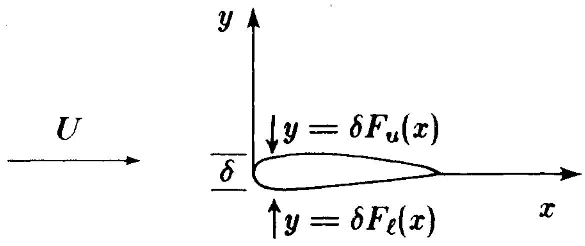

Fig. 1 Flow geometry.

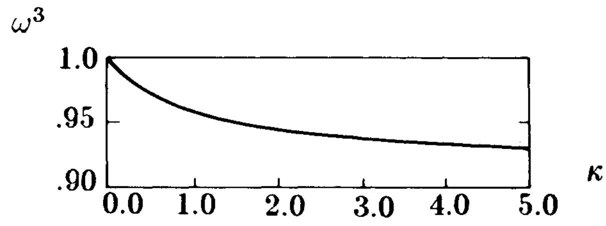

Fig. 2 Transformation parameter.

Then ϕ satisfies

$$[\hat{K} = (\hat{\gamma}+1)\phi_x]\phi_{xx} + \phi_{\tilde{y}\tilde{y}} = 0 \tag{10}$$

and the boundary conditions

$$\phi_{\tilde{y}}(x,0\pm) = F'_{u,\ell}(x), \quad 0<x<1 \tag{11}$$

$$\phi_{x'}\phi_{\tilde{y}} \to 0 \text{ as } x \to -\infty \tag{12}$$

The Kutta condition may be written as

$$[\phi_x]_{\text{TE}} = 0 \tag{13}$$

where TE means trailing edge.

In addition, shock jump conditions must be imposed. If $[\ ]_\delta$ denotes the jump in indicated quantities at the shock surface

$$S(x,\tilde{y}) = x - g(\tilde{y}) = 0 \tag{14}$$

the conditions are

$$\left[\hat{K}\phi_x - \frac{\hat{\gamma}+1}{2}\phi_x^2\right]_\delta - [\phi_{\tilde{y}}]_\delta g'(\tilde{y}) = 0 \tag{15}$$

and

$$[\phi]_\delta = 0 \tag{16}$$

The form of Eq. (10) shows that two flows at different transonic Mach numbers, but with equal values of $\hat{K}$, will be geometrically similar (the main difference being that the "sizes" of the disturbances will be determined by different factors of $\delta^{2/3}$). An analogous similarity law holds for the gas alone, governed by its corresponding similarity parameter.

We wish to see how the changes in the ratio of specific heats and the density ($\gamma \to \hat{\gamma}$, $\rho \to \rho_m$) affect the flow. This will be done by finding a correspondence between the similarity parameters in the two cases. The correspondence is developed by transforming the boundary-value problem [Eq. (10)-(16)] for the dusty gas into a problem for the gas without dust, via group properties of the small-disturbance equation.

Defining

$$\bar{K} \equiv \hat{K}\omega^{-2} \tag{17a}$$

$$\bar{y} \equiv \omega\bar{y} = \bar{\delta}^{1/3} y \tag{17b}$$

$$\bar{\phi} \equiv \omega\phi \tag{17c}$$

where

$$\omega = \left(\frac{\hat{\gamma}+1}{\gamma+1}\right)^{1/3} \text{ and } \bar{\delta} = \omega^3\delta$$

we see that Eqs. (10) and (11) become

$$\bar{K} - (\gamma+1)\bar{\phi}_x\bar{\phi}_{xx} + \bar{\phi}_{\bar{y}\bar{y}} = 0 \tag{18}$$

$$\bar{\phi}_{\bar{y}}(x,0\pm) = F'_{u,\ell}(x), \quad 0<x<1 \tag{19}$$

Furthermore, boundary conditions (11) and (12), and the shock jump conditions (15) and (16), transform into identical conditions for $\bar{\phi}(x,\bar{y};\bar{K})$, simply by replacing symbols via Eqs. (17).

Therefore, under Eqs. (17), the entire boundary-value problem for $\phi(x,\bar{y};\hat{K})$ transforms into the corresponding problem for $\bar{\phi}(x,\bar{y};\bar{K})$; note that the ratio of specific heats for the dusty gas $\hat{\gamma}$ has been replaced by its clear gas counterpart γ. Comparing Eqs. (10) and (18), it is concluded that

$$\phi(x,\bar{y};\hat{K}) = \omega\phi(x,\bar{y};\bar{K}) \tag{20}$$

Now, the similarity parameter $\bar{K}$ for the new problem is of the form

$$\bar{K} = \frac{1-\bar{M}_\infty^2}{\bar{\delta}^{2/3}} = \frac{1-\bar{M}_\infty^2}{(\omega^3\delta)^{2/3}} = \frac{1-\bar{M}_\infty^2}{\omega^2\delta^{2/3}} = \omega^{-2}\hat{K} \tag{21}$$

from which it is concluded that $\bar{M}_\infty = \hat{M}_\infty$. However, $\hat{a}_\infty \neq a_\infty$ implies $\bar{U}_\infty = \bar{M}_\infty a_\infty \neq \hat{M}_\infty \hat{a}_\infty = U_\infty$. Combining Eqs. (20) and (21) gives

$$\phi(x,\bar{y};\hat{K}) = \omega\phi(x,\omega\bar{y};\omega^{-2}\hat{K}) \tag{22}$$

Since ρ_m does not appear in the boundary-value problem [Eqs. (10)-(16)], its affect on the problem is indirect, through the transonic pressure coefficient

$$\hat{C}_p = \frac{p-p_\infty}{\frac{1}{2}\rho_m U_\infty^2 \delta^{2/3}} = -2\phi_x \tag{23}$$

The two pressure coefficients are related by

$$\hat{C}_p = -2\phi_x = -\frac{2}{\omega}\bar{\phi}_x = \frac{\bar{C}_p}{\omega} \tag{24}$$

To complete the details, we note that $\hat{K} = (1-\hat{M}_\infty^2)/\delta^{2/3}$ where $\hat{a}_\infty^2 = \hat{\gamma} a_\infty^2/(\gamma(1+K)$ from Sec. II, so that

$$\hat{K} = \left[1 - \frac{\gamma(1+\kappa)}{\hat{\gamma}} M_\infty^2\right] \Big/ \delta^{2/3} \tag{25}$$

where $\hat{\gamma}$ is given by Eq. (8).

Thus, the small-disturbance problem for a dusty gas is equivalent to one for a clear gas with a modified amplitude of disturbances, a change wing thickness ratio, and a similarity parameter.

The transformation parameter, ω^3, as a function of κ, is shown in Fig. 2 for $c/c_v = 1.0$ which is a representative value for many materials.[4]

These results may be used to predict various aspects of the changes in a flow due to the addition of dust, including the amount of loading necessary for the changes to be significant. For example, for aluminum particles in air where $\rho_p/\rho_g \approx 2000$, the volumetric concentation α must be less than 0.000032 so that the equivalent (clear gas) wing thickness $\bar{\delta}$ differs from δ by less than 1%.

We suggest that a dusty gas may be useful to simulate tests of an airfoil in a clear gas of thickness δ_1 from results in a dusty gas from tests on an airfoil of thickness δ_0. The results presented imply that for a given airfoil of thickness δ_0, the solution in a clear gas for thickness $\delta_1 = \omega^3\delta_0$ at similarity parameter K_1 can be obtained from the solution in a dusty gas with a value of ω equal to $\omega = (\delta_1/\delta_0)^{1/3}$, and similarity parameter $K_0 = \omega^2 K_1$. The loading κ can be obtained from

$$\omega = \left(\frac{\delta_1}{\delta_0}\right)^{1/3} = \left(\frac{\hat{\gamma}+1}{\gamma+1}\right)^{1/3}$$

so that

$$\hat{\gamma} = -1 + \frac{\delta_1}{\delta_0}(\gamma+1)$$

But

$$\hat{\gamma} = \frac{c_p + \kappa c}{c_v + \kappa c} = -1 + \frac{\delta_1}{\delta_0}(\gamma+1)$$

Thus,

$$\kappa = \frac{(\delta_1/\delta_0) - 1}{2 - (\gamma+1)(\delta_1/\delta_0)} \frac{c_p}{c} \tag{26}$$

Thus, for a given gas, and a given dust material (so that c, c_v, c_p, and ρ_p are known), Eq. (26) gives the amount of dust needed to have an airfoil of thickness δ_0 behave as if it were of thickness δ_1. This may offer a savings from the cost of fabrication of several similar airfoils for wind tunnel testing; however, the cost may be offset by the difficulties of using dust in the devices peripheral to the tunnel.

Finally, note that Eqs. (17) enable one to convert a problem involving the flow of an ideal gas with ratio of specific heats $\hat{\gamma}$ into one with $\gamma = 1.4$, whether the change in thermodynamic properties come from adding dust or not. This may be useful in analyzing the flow of gases other than air, allowing such problems to be related back to ones of more familiar airflows.

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